Proposition 4.5.2 (Caplet). *Consider a European call option paying at $T+\tau$ the amount (a* caplet*)*

$$V(T+\tau) = \tau \left(L(T,T,T+\tau) - K\right)^+, \quad \tau > 0.$$

In the Gaussian HJM model (4.40), we have

$$V(t) = P(t,T)\Phi(-d_-) - (1+K\tau)P(t,T+\tau)\Phi(-d_+),$$

where

$$d_\pm = \frac{\ln\left((1+K\tau)P(t,T+\tau)/P(t,T)\right) \pm v/2}{\sqrt{v}},$$
$$v = \int_t^T \left|\sigma_P(u,T+\tau) - \sigma_P(u,T)\right|^2 du.$$

Proof. By definition (4.2) of the Libor rate,

$$V(T+\tau) = \left(\frac{1}{P(T,T+\tau)} - 1 - K\tau\right)^+.$$

As the caplet payoff is $\mathcal{F}_T$-measurable we have

$$V(T) = P(T,T+\tau)V(T+\tau) = \left(1 - (1+K\tau)\,P(T,T+\tau)\right)^+,$$

and we see that the value of the caplet at time T can be written as a scaled payoff of a put option on the zero-coupon bond $P(T,T+\tau)$. Applying Proposition 4.5.1 and call-put parity immediately yields the result. □

Proposition 4.5.3 (Futures Rate). *In the Gaussian HJM model (4.40), futures rates are given by*

$$F(t,T,T+\tau) = \tau^{-1}\left(\left(1/P(t,T,T+\tau)\right)e^{\Omega(t,T)} - 1\right), \qquad (4.42)$$

where

$$\Omega(t,T) = \int_t^T \left[\sigma_P(u,T+\tau) - \sigma_P(u,T)\right]^\top \sigma_P(u,T+\tau)\,du.$$

Proof. From Lemma 4.2.2,

$$\begin{aligned} F(t,T,T+\tau) &= \mathrm{E}_t^{\mathrm{Q}}\left(L(T,T,T+\tau)\right) \\ &= \tau^{-1}\mathrm{E}_t^{\mathrm{Q}}\left(1/P(T,T+\tau) - 1\right) \\ &= \tau^{-1}\mathrm{E}_t^{\mathrm{Q}}\left(G(T) - 1\right), \end{aligned} \qquad (4.43)$$

where we have introduced an auxiliary variable

$$G(t) \triangleq P(t,T)/P(t,T+\tau) = 1/P(t,T,T+\tau).$$