7.6.2.2 A Caveat About the Process Domain

Even though the skew averaging result is obtained in the small slope limit, practical experience validates its broad applicability in option pricing problems. Some typical results can be found in Piterbarg [2005c] and Piterbarg [2006]. Still, the equivalence between the original time-dependent model and the time-averaged one should not be taken too far, as we now proceed to demonstrate. For this, we focus on the simple displaced diffusion model from the previous section, i.e. we consider the time-dependent SDE

$$dS(t) = \lambda \left(b(t)S(t) + (1 - b(t))\, S(0)\right) dW(t), \tag{7.66}$$

and approximate it with

$$dS(t) = \lambda \left(\bar{b}S(t) + \left(1 - \bar{b}\right) S(0)\right) dW(t), \tag{7.67}$$

where $\bar{b}$ is set as in Corollary 7.6.3. While the two SDEs (7.66) and (7.67) may have similar properties in the neighborhood of $S(0)$, they generally do not even have the same range for $S(t)$. For the constant parameter case (7.67) with $\bar{b} > 0$, the process $S(t)$ has a lower bound, the root of the local volatility function: $S(t) \in (S(0)(\bar{b} - 1)/\bar{b}, \infty)$. The same is not necessarily true for the time-dependent SDE (7.66), as should be reasonably clear from the following heuristic argument. If at a given time t, $S(t)$ is close to the root of the local volatility function but still above it, i.e.

$$S(t) \gtrsim S(0)\left(b(t) - 1\right)/b(t),$$

it may so happen that at $t + dt$, $S(t + dt)$ is actually *below* the root of the local volatility function,

$$S\left(t + dt\right) < S(0)\left(b\left(t + dt\right) - 1\right)/b\left(t + dt\right)$$

due to the change in the function $b(\cdot)$. The range

$$\left(-\infty, S(0)\left(b\left(t + dt\right) - 1\right)/b\left(t + dt\right)\right)$$

will then be reachable by $S(\cdot)$. The following proposition provides formal justification.

Proposition 7.6.6. *Consider the SDE*

$$dX(t) = \left(a(t) + b(t)X(t)\right) dW(t) \tag{7.68}$$

with $X(0) \geq -a(0)/b(0)$. *If* $(a(u)/b(u))' \geq 0$ *for all* $u \in [0, t]$, *then* $X(t) > -a(t)/b(t)$ *a.s. If there exists* u, $0 < u < t$, *such that* $(a(u)/b(u))' < 0$, *then* $\mathrm{P}(X(t) < l) > 0$ *for any* $l \in \mathbb{R}$.

Proof. Define

$$\zeta(t) = \int_0^t b(u)\, dW(u) - \frac{1}{2}\int_0^t b^2(u)\, du, \quad Z(t) = \exp\left(\zeta(t)\right).$$

Then the solution to the SDE (7.68) is given by

$$X(t) = Z(t)\left[X(0) - \int_0^t \left(a(u)/b(u)\right) d\left(1/Z(u)\right)\right],$$

as can either be checked directly or obtained from Section 5.6.C of Karatzas and Shreve [1991]. Integrating by parts yields

$$X(t) = Z(t)\left(X(0) + \frac{a(0)}{b(0)}\right) - \frac{a(t)}{b(t)} + Z(t)\int_0^t \frac{\left(a(u)/b(u)\right)'}{Z(u)}\, du.$$

With $X(0) \geq -a(0)/b(0)$,

$$Z(t)\left(X(0) + \frac{a(0)}{b(0)}\right) - \frac{a(t)}{b(t)}$$

is bounded from below by $-a(t)/b(t)$. If $(a(u)/b(u))' \geq 0$ for all $u \in [0, t]$ then the remaining term

$$Z(t)\int_0^t \frac{\left(a(u)/b(u)\right)'}{Z(u)}\, du$$

is non-negative and $X(t)$ is bounded from below by $-a(t)/b(t)$. If, however, there exists u such that $(a(u)/b(u))' < 0$, this term can be arbitrarily negative with positive probability. □

In practice, the likelihood of actually breaching the lower boundary is typically small and we can often safely ignore this possibility. If needed, one can always "regularize" the time-dependent process to limit its range, along the same lines as done in Section 7.2.3.

7.6.3 Skew and Convexity Averaging by Small-Noise Expansion

The technique used in the previous section to derive Proposition 7.6.2 is not the only route to go. An alternative approach relies on *small-noise expansion*, a concept closely related to the Ito-Taylor expansion in Chapter 3. To illustrate the versatility of this method, we shall use it to derive not only the skew averaging result in Corollary 7.6.3, but also to demonstrate how to compute *average convexity* in a time-dependent quadratic model.

As our starting point, we define, for some constant X_0, the quadratic form

$$\begin{aligned}\varphi\left(t, X(t)\right) &= \varphi\left(b(t), c(t), X(t)\right) \\ &= \left(1 - b(t)\right) X_0 + b(t) X(t) + \frac{1}{2} c(t) \left(X(t) - X_0\right)^2,\end{aligned}$$