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# Table of Contents for All Volumes

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## VOLUME I Foundations and Vanilla Models

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### Part I Foundations

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**Part V Risk Management**

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